

Regularization in the gauged Nambu–Jona-Lasinio model

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Abstract

Various prescriptions employed for regulating gauged Nambu–Jona-Lasinio type models such as the top-quark condensate model are discussed. The use of dimensional regularization maintains gauge invariance but destroys the quadratic divergence in the gap equation. If instead a simple ultraviolet momentum cutoff is used to regulate loop integrals, then gauge invariance is destroyed by a quadratically divergent term as well as by ambiguities associated with arbitrary routing of loop momenta. Finally it is shown that one can use dispersion relations to regulate the top-quark condensate model. This prescription maintains gauge invariance and does not depend on arbitrary shifts in loop momenta.

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1 Introduction

In the past few years, following the realization that the top quark is heavier than the gauge boson masses, there has been renewed interest [1] in the original Nambu–Jona-Lasinio (NJL) model [2] to provide a possible dynamical symmetry breaking mechanism for the Standard Model. In particular, new strong forces at a high energy scale are assumed to cause the formation of $\bar{t}t$ bound states (top condensation) and dynamically break the $SU(2) \times U(1)$ symmetry. This leads to an effective low energy theory which is qualitatively equivalent to the Standard Model, but with a heavy top quark playing a direct role in the symmetry breaking.

A minimal scheme implementing the NJL mechanism for the Standard Model was given by Bardeen, Hill and Lindner, (BHL) [3] who obtained precise predictions for the top and Higgs masses. They argued that the low energy effective Lagrangian in the fermionic bubble approximation gives rise to the usual gauge coupling β -functions in the Standard Model from fermion loops. By imposing the compositeness conditions as ultraviolet boundary conditions on the renormalization group flow and assuming the full one-loop β -functions of the Standard Model, BHL found that typically the top quark mass is heavy, $m_{top} \simeq 225 \text{ GeV}$ and $m_{Higgs} \simeq 1.1 m_{top}$, for composite scales ranging from $10^{15} \text{ GeV} \rightarrow 10^{19} \text{ GeV}$. These quantitative predictions are controlled by the infra-red fixed point structure of the renormalization group equations [4] and consequently a heavy top mass is a generic feature of this model.

In order to show that the gauged NJL model is qualitatively equivalent to the Standard Model one must take care in choosing a consistent regularization scheme. This is because the four-Fermi interaction in the NJL model leads to quadratically divergent terms which can destroy many of the desired qualitative features of the Standard Model, before any quantitative predictions can be made.

In this work we discuss various regularization schemes one can implement in the gauged NJL models such as the top-quark condensate model [3]. In these models the dynamics leads to a gap equation, where unlike in the original BCS theory, there are quadratically divergent terms. While these terms lead to fine tuning problems, they are absolutely necessary for any consistent model and the choice of regularization scheme must be made compatible with them. In order to make contact with the Standard Model, the vector gauge boson masses in the NJL model must be induced in a locally gauge invariant manner. One common scheme often employed is dimensional regularization and we will discuss the consequences of implementing this scheme consistently for the NJL model. In addition Willey [5] recently argued that the NJL model suffers from inherent ambiguities arising from the arbitrary routing of momenta through quadratically divergent fermion bubble graphs. Willey employed an ultraviolet momentum cutoff for all integrals and we will examine the consequences of these ambiguities for the top-quark condensate model. Finally we will show that a consistent prescription for regulating the top-quark condensate model is to use dispersion relations. This prescription is gauge invariant and avoids the problems of ambiguities arising from quadratic divergences.

2 The NJL model and dimensional regularization

Let us begin with the minimal top-condensate model of Bardeen, Hill and Lindner [3]. The NJL Lagrangian for the Standard Model at the scale Λ is given by

$$\mathcal{L} = \mathcal{L}_{kinetic} + G(\bar{\Psi}_{La}t_R^a)(\bar{t}_{Rb}\Psi_L^b) \quad ; \quad \Psi_L = \begin{pmatrix} t_L \\ b_L \end{pmatrix} \quad (1)$$

where a and b are colour indices and $\mathcal{L}_{kinetic}$ contains only the gauge boson and fermion kinetic energy terms. Note that (1) is an effective Lagrangian of some renormalizable interaction above the cutoff scale Λ and that the four-fermion coupling constant, G must be positive to ensure an attractive interaction. The induced top quark mass resulting from the four-fermion interaction in (1) is obtained by solving the gap equation to leading order in N_{colour}

$$1 = 2GN_c i \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m_t^2} . \quad (2)$$

Assuming that (1) describes physics up to the scale Λ the integral (2) can be regularized with an ultraviolet Euclidean momentum cutoff Λ . This leads to the gap equation

$$1 = \frac{GN_c}{8\pi^2} \left(\Lambda^2 - m_t^2 \ln \frac{\Lambda^2}{m_t^2} \right) , \quad (3)$$

where we have only kept leading order terms. Notice that there is a quadratically divergent term in the gap equation (3) which is nothing other than the reincarnation of the Higgs boson mass quadratic divergence in the Standard Model and leads to the usual gauge-hierarchy problem.

The low energy effective Lagrangian (1) is clearly gauge invariant because it only contains the gauge boson kinetic energy. When the scalar $\bar{t}t$ condensate forms, the gauge bosons must acquire mass in a $SU(2) \times U(1)$ gauge invariant manner because an explicit mass term would break gauge invariance. In the top-quark condensate model the gauge boson mass arises from the fermion mass dependence of the gauge boson self-energy, and so one cannot simply discard all the non-gauge invariant terms in the self-energy. If one employs dimensional regularization then it turns out that only the fermion mass dependent non-gauge invariant terms survive and they combine with terms from the Nambu-Goldstone mode contributions, so that the gauge bosons become massive in a gauge invariant manner. However the unfortunate feature of the dimensional regularization scheme is that it also kills all quadratic divergences. This is because in dimensional regularization

$$\lim_{m^2 \rightarrow 0} \int \frac{d^Dk}{k^2 - m^2} = 0 . \quad (4)$$

While this is fine for the quadratically divergent non-gauge invariant terms in the gauge boson self-energy, it is a disaster for the gap equation because dimensionally

regularizing the integral in (2) destroys the quadratic divergence in (3). In particular the gap equation without the quadratically divergent term is given by

$$1 = -\frac{GN_c}{8\pi^2}m_t^2 \left(\ln \frac{\mu^2}{m_t^2} + 1 \right), \quad (5)$$

where μ is the renormalization scale. Eq. (5) would mean that the four-Fermi coupling $G < 0$, corresponding to a repulsive interaction. This would contradict the assumed attractive interaction between top quarks in (1). Clearly we need a regulator that is sensitive to the quadratic divergences.

3 Ambiguities in the NJL model

The simplest prescription for regulating divergent integrals that does not destroy the quadratic divergences in the gap equation is to use an ultraviolet Euclidean momentum cutoff Λ . Let us now examine the consequences of using this simple prescription for the top-condensate model. Consider the sum of fermion-fermion scattering amplitudes depicted in Figure 1, where the amplitudes are summed to leading order in N_{colour} and the QCD coupling constant is zero (fermion bubble approximation). Assuming that the gap equation is satisfied, the fermion-fermion scattering amplitude in the scalar $\bar{t}t$ channel is given by

$$\begin{aligned} \Gamma_s(p^2) &= \frac{iG}{2} \left(1 - \frac{iG}{2} N_c \int \frac{d^4l}{(2\pi)^4} \text{Tr} \left[\frac{1}{\not{l} + (1-\alpha)\not{p} - m_t} \frac{1}{\not{l} - \alpha\not{p} - m_t} \right] \right)^{-1} \\ &\simeq \frac{(4\pi)^2}{2N_c} \left(\frac{1}{2} \Delta(\alpha) p^2 + (p^2 - 4m_t^2) \int_0^1 dx \ln \frac{\Lambda^2}{m_t^2 - p^2 x(1-x)} \right)^{-1}, \end{aligned} \quad (6)$$

where α is an arbitrary routing parameter introduced in the loop momentum (see Figure 2) and $\Delta(\alpha) = (1 - 2\alpha + 2\alpha^2)$. Normally the parameter α is neglected because the momentum integration variable can be shifted by a finite amount for integrals which are at most logarithmically divergent. However for integrals which are more divergent, this shift of integration variable results in the appearance of an extra “surface term” which depends on α [6]. This causes the pole in the scattering amplitude (6) to shift by an amount proportional to $\Delta(\alpha)/\ln(\Lambda^2/m_t^2)$ from the usual value at $p^2 = 4m_t^2$. However since Λ is necessarily finite and α is an unrestricted constant, the

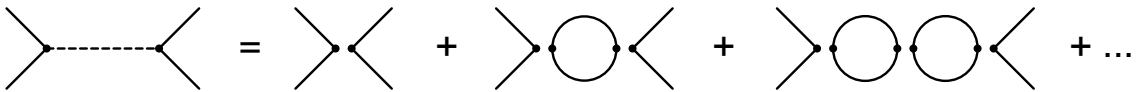


Figure 1: *The infinite sum of fermion bubble diagrams. The scalar $\bar{t}t$ channel has a pole at $2m_t$ which corresponds to the composite Higgs scalar. The pseudoscalar $\bar{t}t$ and flavoured $t\bar{b}$ channels give rise to three massless Nambu-Goldstone bosons.*

pole receives an arbitrary correction and so makes the relation between the top and physical Higgs boson mass arbitrary. Note that when there is no routing ambiguity or $\alpha = 0$, the correction to the pole term is negligible for large Λ .

This momentum routing ambiguity, in the context of the NJL model, was first noticed by Willey [5] and is the result of using a momentum cutoff regulator. It should be remarked that this routing ambiguity is completely different from an earlier ambiguity noted by Hasenfratz et al. [7], who considered a lattice formulation of the NJL theory. In the lattice theory, higher derivative interaction terms can be added to the NJL Lagrangian, which are in equivalent universality classes, in the sense of defining the theory by going to a critical point. These terms introduce additional arbitrary constants and give an ambiguous prediction for the ratio m_σ^2/m_f^2 . However assuming that the non-renormalizable four-Fermi interaction arises from a renormalizable gauge theory at high energy, it can be shown that the additional arbitrary constants associated with higher derivative interaction terms are numbers of $\mathcal{O}(1)$ and do not greatly influence the BHL predictions [8].

Similarly in the neutral pseudoscalar $\bar{t}t$ channel the arbitrary surface term, $\Delta(\alpha)$ will appear. The scattering amplitude is given by

$$\Gamma_P(p^2) \simeq \frac{(4\pi)^2}{2N_c p^2} \left(\frac{\Delta(\alpha)}{2} + \int_0^1 dx \ln \frac{\Lambda^2}{m_t^2 - p^2 x(1-x)} \right)^{-1}, \quad (7)$$

where the gap equation has again been invoked. Notice that the neutral Nambu-Goldstone pole still occurs at $p = 0$, except that now the coupling constant depends on the arbitrary parameter α . A similar dependence on the arbitrary parameter α occurs for the charged Nambu-Goldstone modes when one performs the fermion bubble sum in the flavoured tb channel. Assuming $m_b \simeq 0$ one obtains for the scattering amplitude in the tb channel

$$\Gamma_F(p^2) \simeq \frac{(4\pi)^2}{8N_c p^2} \left(\frac{\Delta(\alpha)}{4} + \int_0^1 dx (1-x) \ln \frac{\Lambda^2}{m_t^2(1-x) - p^2 x(1-x)} \right)^{-1}. \quad (8)$$

Again notice that the routing ambiguity affects only the coupling constant and not the masslessness of the charged Nambu-Goldstone boson.

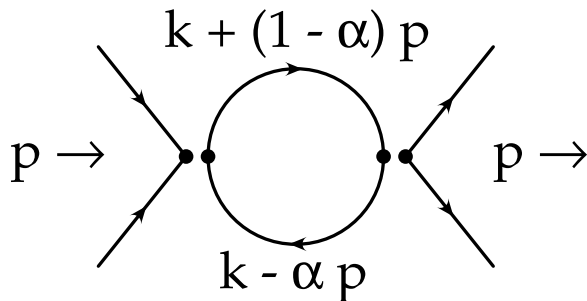


Figure 2: The loop momentum routing in a fermion bubble diagram.

The gauge bosons in the NJL model become massive by “eating” up the dynamically generated Nambu-Goldstone modes. This corresponds to a dynamical Higgs mechanism. Since there are routing ambiguities in the Nambu-Goldstone modes it is clear that the gauge boson masses will also be afflicted with this arbitrariness. The question of whether gauge invariance can survive this arbitrariness needs to be checked.

Consider first the W-boson propagator, where the gauge fields are rescaled so that the kinetic energy is $-1/(4g^2)F^{\mu\nu}F_{\mu\nu}$. In the fermion bubble approximation, corrections to the propagator arise from the diagrams depicted in Figure 3. Again we will choose to regularize all divergent integrals with an ultraviolet cutoff, Λ and introduce an arbitrary parameter, α to represent the routing ambiguity in the loop momentum. The W-boson inverse propagator is given by the sum of three terms

$$\frac{1}{g_2^2}D_{\mu\nu}^W(p)^{-1} = \frac{1}{g_2^2}(p_\mu p_\nu - g_{\mu\nu}p^2) + \Pi_{\mu\nu}(p) - \frac{1}{8}J_\mu(p)\Gamma_F(p^2)J_\nu(p), \quad (9)$$

where g_2 is the SU(2) coupling constant and

$$\Pi_{\mu\nu}(p) = \frac{i}{8} \int \frac{d^4l}{(2\pi)^4} \text{Tr} \left[\gamma_\mu(1 - \gamma_5) \frac{1}{\not{l} + (1 - \alpha)\not{p} - m_b} \gamma_\nu(1 - \gamma_5) \frac{1}{\not{l} - \alpha\not{p} - m_t} \right], \quad (10)$$

$$J_\mu(p) = i \int \frac{d^4l}{(2\pi)^4} \text{Tr} \left[\gamma_\mu(1 - \gamma_5) \frac{1}{\not{l} + (1 - \alpha)\not{p} - m_b} (1 + \gamma_5) \frac{1}{\not{l} - \alpha\not{p} - m_t} \right]. \quad (11)$$

In Eq.(9) $\Pi_{\mu\nu}$ is the usual one loop gauge boson self-energy. There will be two types of divergent integrals appearing in (10) and (11) which are not invariant under an arbitrary shift of the loop momentum. These are given by

$$\int \frac{d^4l}{(2\pi)^4} \frac{l_\mu}{[(l + \chi p)^2 - M^2]^2} = \int \frac{d^4l}{(2\pi)^4} \frac{-\chi p_\mu}{(l^2 - M^2)^2} + \frac{i}{32\pi^2} \chi p_\mu, \quad (12)$$

$$\begin{aligned} \int \frac{d^4l}{(2\pi)^4} \frac{l_\mu l_\nu}{[(l + \chi p)^2 - M^2]^2} &= \int \frac{d^4l}{(2\pi)^4} \frac{(l_\mu l_\nu + \chi^2 p_\mu p_\nu)}{(l^2 - M^2)^2} + \frac{i}{96\pi^2} \chi^2 (g_{\mu\nu} p^2 - p_\mu p_\nu) \\ &+ \frac{i}{4\pi^2} \chi^4 p_\mu p_\nu \quad p^2 \quad \int_0^1 dy \int_0^1 dz \frac{y^3 z^2}{M^2 - \chi^2 p^2 y(1 - y + yz(1 - z))}, \end{aligned} \quad (13)$$

where χ is an l and p independent constant. Using these expressions and assuming the top quark mass satisfies the gap equation (3) the W-boson vacuum polarization

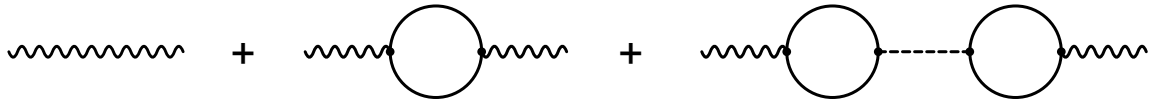


Figure 3: *Diagrams contributing to the gauge boson vacuum polarization in the fermion bubble approximation.*

becomes

$$\frac{1}{g_2^2} D_{\mu\nu}^W(p)^{-1} = \left(\frac{p_\mu p_\nu}{p^2} - g_{\mu\nu} \right) \left[\frac{p^2}{\bar{g}_2^2(p^2, \alpha)} - \bar{f}^2(p^2, \alpha) \right] - g_{\mu\nu} \bar{h}^2(p^2, \alpha), \quad (14)$$

where

$$\frac{1}{\bar{g}_2^2(p^2, \alpha)} = \frac{1}{g_2^2} + \frac{N_c}{(4\pi)^2} \left[\int_0^1 dx \, 2x(1-x) \ln \frac{\Lambda^2}{M^2(x)} - \frac{13}{18} + \frac{5}{3}\alpha(1-\alpha) - 8B(\alpha, M) \right], \quad (15)$$

$$\bar{f}^2(p^2, \alpha) = m_t^2 \frac{N_c}{(4\pi)^2} \frac{\left(\int_0^1 dx \, (1-x) \ln \frac{\Lambda^2}{M^2(x)} - \frac{1}{2}(\frac{1}{2} + \alpha) \right)^2}{\int_0^1 dx \, (1-x) \ln \frac{\Lambda^2}{M^2(x)} - \frac{1}{2}(\frac{1}{2} + \alpha - \alpha^2)}, \quad (16)$$

$$\bar{h}^2(p^2, \alpha) = \frac{N_c}{(4\pi)^2} \frac{1}{2} \left[\Lambda^2 - m_t^2 A(\alpha, M) + p^2 \left(\frac{1}{3} - \alpha + \alpha^2 + 8B(\alpha, M) \right) \right], \quad (17)$$

$$A(\alpha, M) = \frac{\alpha(1+\alpha) \int_0^1 dx \, (1-x) \ln \frac{\Lambda^2}{M^2(x)} - \frac{1}{4}\alpha(1+3\alpha)}{\int_0^1 dx \, (1-x) \ln \frac{\Lambda^2}{M^2(x)} - \frac{1}{2}(\frac{1}{2} + \alpha - \alpha^2)}, \quad (18)$$

$$B(\alpha, M) = p^2 \int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{(x-\alpha)^4 y^3 z^2}{M^2(x) - (x-\alpha)^2 p^2 y(1-y+yz(1-z))}, \quad (19)$$

and $M^2(x) = m_t^2(1-x) - p^2 x(1-x)$. The expression for the W-boson inverse propagator (14) contains a quadratic divergence ($\propto g_{\mu\nu} \Lambda^2$) which does not appear in the corresponding expression from BHL [3]. This stems from the fact that an ultraviolet cutoff has been used instead of dimensional regularization. This quadratic term destroys gauge invariance and was first noticed a long time ago by Wentzel [9]. Apart from this quadratically divergent term, which would appear irrespective of loop momentum ambiguities, note that the arbitrary constant α is enough by itself to destroy gauge invariance. Unfortunately the ambiguity in the one loop W-boson self-energy and the Nambu-Goldstone boson term do not combine together to save gauge invariance.

Similarly for the neutral gauge bosons we consider the contributions arising from the diagrams in Figure 3 and include all surface terms arising from divergent integrals. Working in the $(B - W^3)$ basis we obtain in the limit $m_b \simeq 0$

$$\begin{aligned} \frac{1}{g_i g_j} D_{\mu\nu}^0(p)^{-1} &= \left(\frac{p_\mu p_\nu}{p^2} - g_{\mu\nu} \right) \left(\left[\begin{array}{cc} \frac{1}{g_1^2(p^2, \alpha)} & 0 \\ 0 & \frac{1}{g_2^2(p^2, \alpha)} \end{array} \right] p^2 - \left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right] f^2(p^2, \alpha) \right) \\ &\quad - g_{\mu\nu} \left(\left[\begin{array}{cc} \frac{20}{9} & 0 \\ 0 & \frac{4}{3} \end{array} \right] Q_t + \left[\begin{array}{cc} \frac{2}{9} & 0 \\ 0 & \frac{2}{3} \end{array} \right] Q_b - \left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right] h^2(p^2, \alpha) \right) \end{aligned} \quad (20)$$

where g_1 is the U(1) gauge coupling constant, g_2 is the SU(2) gauge coupling constant and

$$\begin{aligned} \frac{1}{g_1^2(p^2, \alpha)} &= \frac{1}{g_1^2} + \frac{N_c}{(4\pi)^2} \left[\int_0^1 dx \, x(1-x) \left(\frac{20}{9} \ln \frac{\Lambda^2}{M_t^2(x)} + \frac{2}{9} \ln \frac{\Lambda^2}{M_b^2(x)} \right) \right. \\ &\quad \left. - \frac{11}{9} \left(\frac{13}{18} - \frac{5}{3}\alpha(1-\alpha) \right) - \frac{80}{9} B(\alpha, M_t) - \frac{8}{9} B(\alpha, M_b) \right] \end{aligned} \quad (21)$$

$$\frac{1}{g_2^2(p^2, \alpha)} = \frac{1}{g_2^2} + \frac{N_c}{(4\pi)^2} \left[\int_0^1 dx x(1-x) \left(\frac{4}{3} \ln \frac{\Lambda^2}{M_t^2(x)} + \frac{2}{3} \ln \frac{\Lambda^2}{M_b^2(x)} \right) - \frac{13}{18} + \frac{5}{3} \alpha(1-\alpha) - \frac{16}{3} B(\alpha, M_t) - \frac{8}{3} B(\alpha, M_b) \right], \quad (22)$$

$$f^2(p^2, \alpha) = \frac{N_c}{(4\pi)^2} \left[\frac{p^2}{3} \left(\int_0^1 dx x(1-x) \ln \frac{M_b^2(x)}{M_t^2(x)} - 4B(\alpha, M_t) + 4B(\alpha, M_b) \right) + \frac{1}{2} m_t^2 \frac{(\int_0^1 dx \ln \frac{\Lambda^2}{M_t^2(x)} - 1)^2}{\int_0^1 dx \ln \frac{\Lambda^2}{M_t^2(x)} + \frac{1}{2} \Delta(\alpha) - 1} \right], \quad (23)$$

$$Q_i(p^2, \alpha, m_i) = \frac{N_c}{(4\pi)^2} \frac{1}{4} \left[\Lambda^2 - m_i^2 + p^2 \left(\frac{1}{3} - \alpha + \alpha^2 + 8B(\alpha, M_i) \right) \right], \quad (24)$$

$$h^2(p^2, \alpha) = \frac{N_c}{(4\pi)^2} \left[\frac{m_t^2}{4} \left(\frac{(\Delta(\alpha) - \frac{1}{3}) \int_0^1 dx \ln \frac{\Lambda^2}{M_t^2(x)} + \frac{1}{3} - \frac{7}{6} \Delta(\alpha)}{\int_0^1 dx \ln \frac{\Lambda^2}{M_t^2(x)} + \frac{1}{2} \Delta(\alpha) - 1} \right) + \frac{2}{3} p^2 (B(\alpha, M_t) - B(\alpha, M_b)) \right], \quad (25)$$

with $M_t^2(x) = m_t^2 - p^2 x(1-x)$ and $M_b^2(x) = m_b^2 - p^2 x(1-x)$. Again we note that gauge invariance is destroyed by ambiguities coming from the surface terms of the linearly and quadratically divergent integrals as well as from the Wentzel term. In the limit that all surface terms are zero and ignoring the quadratic divergence arising from the cutoff regulator, the above expressions reduce to the results of BHL [3]. Thus by regulating divergent loop integrals with an ultraviolet momentum cutoff, one finds that the gauged NJL model is plagued with the problems of a non gauge-invariant Higgs mechanism and ambiguous quantitative predictions resulting from arbitrary loop momentum routing. These problems are solely an artifact of using a simple momentum cutoff.

4 Dispersion relations

An alternative prescription for regulating the top-quark condensate model is to use dispersion relations. This prescription was previously employed by Nambu and Jona-Lasinio [2] in their dynamical model of elementary particles. In this prescription one first calculates the imaginary part of an amplitude \mathcal{A} by means of Cutkosky's rule [10] and then forms the complete amplitude by use of the unsubtracted dispersion relation

$$\mathcal{A}(p^2) = \frac{1}{\pi} \int_L d\kappa^2 \frac{\text{Im} \mathcal{A}(\kappa^2)}{\kappa^2 - p^2 - i\epsilon}, \quad (26)$$

where L is some cut along the real axis and the surface term at infinity has been neglected by assumption. In general the high energy behaviour of \mathcal{A} makes the right-hand side of (26) divergent. This ultraviolet divergence is regulated by replacing the

upper limit of integration by a cutoff scale $4\Lambda^2$, which represents the maximum total energy squared. For energies greater than this scale $\text{Im}\mathcal{A} = 0$.

The first immediate consequence of the dispersion relation (26) is that quadratically divergent amplitudes are no longer dependent on an arbitrary parameter α resulting from loop momenta shifts. This is because to calculate the imaginary part of a one loop amplitude, for example, the intermediate particle states must be put on-shell according to Cutkosky's rule. As a result the arbitrary dependence cancels out and quantitative predictions will not be jeopardized.

In the NJL model the fermion mass satisfies a gap equation and so we need to reformulate this self consistent condition using dispersion relations. Consider first the sum of fermion bubbles in the pseudoscalar $\bar{t}t$ channel (γ_5). The scattering amplitude is given by

$$\Gamma_P(p^2) = \frac{G}{2} \frac{1}{1 - J_P(p^2)}, \quad (27)$$

where

$$J_P(p^2) = \frac{GN_c}{(4\pi)^2} \int_{4m_t^2}^{4\Lambda^2} d\kappa^2 \frac{\kappa^2}{\kappa^2 - p^2 - i\epsilon} \sqrt{1 - \frac{4m_t^2}{\kappa^2}}. \quad (28)$$

Requiring $J_P(0) = 1$ or

$$1 = \frac{GN_c}{(4\pi)^2} \int_{4m_t^2}^{4\Lambda^2} d\kappa^2 \sqrt{1 - \frac{4m_t^2}{\kappa^2}}, \quad (29)$$

leads to a pole at $p^2 = 0$, which is the massless neutral Nambu-Goldstone mode. Evaluating the integral in (29) to leading order gives rise to the self consistent condition

$$1 = \frac{GN_c}{8\pi^2} \left(2\Lambda^2 - m_t^2 \ln \frac{\Lambda^2}{m_t^2} \right). \quad (30)$$

This is equivalent to the gap equation obtained earlier, (3), by noting that the ultraviolet Euclidean momentum cutoff (Λ_E) used in sections 2 and 3 is related to the dispersion integral cutoff by $\Lambda_E = \sqrt{2}\Lambda$.

Similarly for the scattering amplitude in the scalar $\bar{t}t$ channel we obtain

$$\Gamma_S(p^2) = -\frac{G}{2} \frac{1}{1 - J_S(p^2)}, \quad (31)$$

where

$$J_S(p^2) = \frac{GN_c}{(4\pi)^2} \int_{4m_t^2}^{4\Lambda^2} d\kappa^2 \frac{\kappa^2 - 4m_t^2}{\kappa^2 - p^2 - i\epsilon} \sqrt{1 - \frac{4m_t^2}{\kappa^2}}. \quad (32)$$

If we invoke the condition, (29) then $J_S(4m_t^2) = 1$ and Γ_S will have a pole at $p^2 = 4m_t^2$. This is the dynamically generated scalar bound state or Higgs mode.

The remaining flavoured tb channels will similarly give rise to

$$\Gamma_F(p^2) = -\frac{G}{4} \frac{1}{1 - J_F(p^2)}, \quad (33)$$

where

$$J_F(p^2) = \frac{GN_c}{(4\pi)^2} \int_{m_t^2}^{4\Lambda^2} d\kappa^2 \frac{\kappa^2}{\kappa^2 - p^2 - i\epsilon} \left(1 - \frac{m_t^2}{\kappa^2}\right)^2, \quad (34)$$

and the bottom mass has been neglected ($m_b = 0$). Again invoking the self-consistent condition, (29) leads to the massless charged Nambu-Goldstone modes ($J_F(0) = 1$). Thus in the dispersion relation approach the gap equation is replaced by the condition requiring massless Nambu-Goldstone modes.

In the absence of spontaneous symmetry breaking, on-shell quantities are always gauge invariant. When the symmetry is spontaneously broken we need to check that the gauge bosons receive their mass in a gauge invariant manner. As depicted in Figure 3, the gauge bosons acquire their mass by absorbing the massless Nambu-Goldstone modes. In particular, evaluating the W-boson vacuum polarization diagrams using dispersion relations, one obtains for the W-boson inverse propagator

$$\frac{1}{g_2^2} D_{\mu\nu}^W(p)^{-1} = \left(\frac{p_\mu p_\nu}{p^2} - g_{\mu\nu} \right) \left(\frac{p^2}{\bar{g}_2^2(p^2)} - \bar{f}^2(p^2) \right), \quad (35)$$

where the dispersion integrals for \bar{g}_2^2 and \bar{f}^2 are

$$\frac{1}{\bar{g}_2^2(p^2)} = \frac{1}{g_2^2} + \frac{1}{3} \frac{N_c}{(4\pi)^2} \int_{m_t^2}^{4\Lambda^2} d\kappa^2 \frac{1}{\kappa^2 - p^2 - i\epsilon} \left(1 + \frac{2m_t^2}{\kappa^2}\right) \left(1 - \frac{m_t^2}{\kappa^2}\right)^2, \quad (36)$$

$$\bar{f}^2(p^2) = \frac{m_t^2}{2} \frac{N_c}{(4\pi)^2} \int_{m_t^2}^{4\Lambda^2} d\kappa^2 \frac{1}{\kappa^2 - p^2 - i\epsilon} \left(1 - \frac{m_t^2}{\kappa^2}\right)^2. \quad (37)$$

Note that the W-boson inverse propagator is transverse and corresponds to a gauge invariant Higgs mechanism. The induced W-boson mass appears as a pole in the propagator and is a solution of the equation

$$\frac{M_W^2}{\bar{g}_2^2(M_W^2)} - \bar{f}^2(M_W^2) = 0. \quad (38)$$

To leading order in Λ and assuming $M_W \ll m_t$, the W-boson mass is given by

$$M_W^2 = g_2^2 \frac{N_c}{(4\pi)^2} \frac{m_t^2}{2} \log \frac{\Lambda^2}{m_t^2}, \quad (39)$$

which is similar to the mass relation derived by BHL. As an aside we note that (39) is the same as the one loop correction to the W-boson mass in the Standard Model with an elementary Higgs scalar. The effective top Yukawa coupling, f_t obtained from (31) is

$$\frac{1}{f_t^2} = \frac{N_c}{(4\pi)^2} \log \frac{\Lambda^2}{m_t^2}, \quad (40)$$

and this is easily seen to be compatible with (39), where $M_W^2 = \frac{1}{2} g_2^2 m_t^2 / f_t^2$.

Finally for completeness we give the dispersion integrals for the neutral gauge boson propagators. Working in the $(B - W^3)$ basis leads to

$$\frac{1}{g_i g_j} D_{\mu\nu}^0(p)^{-1} = \left(\frac{p_\mu p_\nu}{p^2} - g_{\mu\nu} \right) \left(\left[\begin{array}{cc} \frac{1}{g_1^2(p^2)} & 0 \\ 0 & \frac{1}{g_2^2(p^2)} \end{array} \right] p^2 - \left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right] f^2(p^2) \right), \quad (41)$$

where

$$\frac{1}{g_1^2(p^2)} = \frac{1}{g_1^2} + \frac{20}{9} \mathcal{J}(m_t^2, p^2, \Lambda^2) + \frac{2}{9} \mathcal{J}(m_b^2, p^2, \Lambda^2), \quad (42)$$

$$\frac{1}{g_2^2(p^2)} = \frac{1}{g_2^2} + \frac{4}{3} \mathcal{J}(m_t^2, p^2, \Lambda^2) + \frac{2}{3} \mathcal{J}(m_b^2, p^2, \Lambda^2), \quad (43)$$

$$f^2(p^2) = \frac{1}{3} p^2 \mathcal{J}(m_t^2, p^2, \Lambda^2) - \frac{1}{3} p^2 \mathcal{J}(m_b^2, p^2, \Lambda^2) + m_t^2 \mathcal{K}(m_t^2, p^2, \Lambda^2), \quad (44)$$

and the dispersion integrals are given by

$$\mathcal{J}(m^2, p^2, \Lambda^2) = \frac{1}{6} \frac{N_c}{(4\pi)^2} \int_{4m^2}^{4\Lambda^2} d\kappa^2 \frac{1}{\kappa^2 - p^2 - i\epsilon} \left(1 + \frac{2m^2}{\kappa^2} \right) \sqrt{1 - \frac{4m^2}{\kappa^2}}, \quad (45)$$

$$\mathcal{K}(m^2, p^2, \Lambda^2) = \frac{1}{2} \frac{N_c}{(4\pi)^2} \int_{4m^2}^{4\Lambda^2} d\kappa^2 \frac{1}{\kappa^2 - p^2 - i\epsilon} \sqrt{1 - \frac{4m^2}{\kappa^2}}. \quad (46)$$

It is apparent from (41) that the neutral gauge boson will become massive in a gauge invariant manner. Thus in the case of spontaneous symmetry breaking Eqs. (35) and (41) show that gauge invariance is saved by the Nambu-Goldstone mechanism when dispersion relations are used.

The high energy renormalization group running of the SU(2) and U(1) gauge coupling constants can be obtained from the dispersion integral expressions (36), (42) and (43). Assuming that $p^2 \gg m_t^2$ one obtains

$$16\pi^2 p^2 \frac{d}{dp^2} \frac{1}{g_1^2(p^2)} = -\frac{11}{27} N_c, \quad (47)$$

and

$$16\pi^2 p^2 \frac{d}{dp^2} \frac{1}{\bar{g}_2^2(p^2)} = 16\pi^2 p^2 \frac{d}{dp^2} \frac{1}{g_2^2(p^2)} = -\frac{N_c}{3}. \quad (48)$$

Note that the SU(2) gauge coupling running obtained from the charged gauge boson and neutral gauge boson expressions have identical high energy running as expected.

Let us now compare Eqs.(47) and (48) with the usual β -functions. The Standard Model one-loop β -functions for the SU(2) and U(1) gauge coupling constants are given by

$$16\pi^2 \frac{dg_1}{dt} = \left[\frac{1}{6} + \frac{11}{27} N_c n_q + n_l \right] g_1^3, \quad (49)$$

$$16\pi^2 \frac{dg_2}{dt} = \left[-\frac{43}{6} + \frac{N_c}{3} n_q + \frac{1}{3} n_l \right] g_2^3, \quad (50)$$

where $n_q(n_l)$ are the number of quark (lepton) generations. One can see that the single generation quark loop contributions to the Standard Model β -functions agree with the high energy running (Eqs.(47) and (48)) obtained in the fermion bubble approximation from the dispersion integral expressions for the gauge coupling constants. This agreement further establishes the consistent equivalence between the minimal top-condensate model and the Standard Model when dispersion relations are used.

Clearly, in order to go beyond the fermion bubble approximation in the top-condensate model one can utilize the full one-loop β -functions of the Standard Model to obtain definite predictions for the top and Higgs mass as originally advocated by Bardeen, Hill and Lindner [3].

5 Conclusion

The choice of a consistent regulator in the gauged NJL model is important in order to establish a qualitative equivalence with the Standard Model. The use of an ultraviolet Euclidean momentum cutoff Λ to regulate divergent loop integrals is not a good regularization prescription for the gauged NJL model. This is because the prescription suffers from ambiguities associated with quadratic divergences, which destroy the gauge invariance of the theory. The dependence on the arbitrary parameter α also means that no quantitative conclusions could be drawn from the model if this regulator were used. In addition, the simple cutoff prescription produces the well known quadratic divergence ($\propto g_{\mu\nu}\Lambda^2$) which destroys gauge invariance. An alternative prescription would be to use dimensional regularization which would eliminate this quadratically divergent term and give rise to a gauge invariant Higgs mechanism. However the quadratic divergences in the gap equation would also be destroyed, leading to an inconsistent theory.

A more suitable regularization prescription for the top-quark condensate model that maintains gauge invariance without destroying the quadratic terms is to use unsubtracted dispersion relations. In this case the gap equation is equivalent to the condition requiring the masslessness of the Nambu-Goldstone bosons. Arbitrary shifts in the loop momenta do not affect quantitative predictions because intermediate particle states are put on-shell when the imaginary part of an amplitude is calculated. Thus provided the gap equation is satisfied, the dynamical Higgs mechanism is gauge invariant with no corresponding routing ambiguities. If the high energy running of the gauge coupling constants is obtained from the dispersion integral expressions in the fermion bubble approximation then the usual fermion loop contributions to the Standard Model β -functions are obtained. This consistently shows that the top-quark condensate model is equivalent to Standard Model when dispersion relations are used. The BHL predictions for the top and Higgs mass are then obtained by employing the full one-loop β -functions of the Standard Model and imposing the compositeness conditions of the gauged NJL model.

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